

Geometrical Issues for the 3-dim Quantum Euclidean Space *

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Abstract

We briefly describe our application of a version of noncommutative differential geometry to the 3-dim quantum space covariant under the quantum group of rotations $SO_q(3)$ and sketch how this might be used to determine the correct physical interpretation of the geometrical observables.

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1 Introduction and preliminaries

It is a rather old idea that the micro-structure of space-time at the Planck level might be better described using a noncommutative geometry. An interesting problem is that of finding an appropriate noncommutative version of Minkowski space (see e.g. [19, 1]). Here we consider a noncommutative generalization [7] of the Cartan ‘moving-frame’ formalism which provides in certain special cases an interesting bridge between the ‘Dirac-operator’ formalism of Connes [5] and the quantum-group formalism of Woronowicz [20]. We apply it to the quantum Euclidean space \mathbf{R}_q^3 [8], namely the quantum space covariant under the quantum group $SO_q(3)$. We first outline the main results of a previous article [11] and then sketch how these results can be used to extract informations on the geometrical structure of \mathbf{R}_q^3 . For the generalizations of these results to the quantum Euclidean spaces of higher dimensions see Ref. [3] and the article entitled “Geometrical Techniques on the N -dimensional Quantum Euclidean Spaces” in the present proceedings. In Ref. [11] we introduced a metric and an ‘almost’ metric-compatible linear connection on the quantum Euclidean space, equipped with its (two) standard $SO_q(3)$ -covariant differential calculi; correspondingly, the ‘frame’ or dreibein has been also found. Modulo a conformal factor, which might however be reabsorbed into a formulation of the metric compatibility more suitable for the present case, the curvature turns out to be zero, suggesting that the quantum space is flat as in the commutative limit. This is of course welcome if the postulated noncommutative algebra and differential calculus are really to describe flat Euclidean space \mathbf{R}^3 in the commutative limit. In a separate paper we shall show that in the same limit the traditional quantum space coordinates go to suitable general (non-cartesian) coordinates. This will allow to cure some unpleasant features [12] of a naive physical interpretation of the representation theory of the algebra of function on \mathbf{R}_q^3 . Using the properties of the volume form on \mathbf{R}_q^3 , here we just give some arguments why this phenomenon must occur.

The preliminaries contained in this section are a variation of noncommutative geometry which has been proposed [6, 7] as a noncommutative version of the Cartan ‘moving-frame’ calculus. The starting point is a noncommutative algebra \mathcal{A} with unit which has as commutative limit the algebra of functions on some manifold \mathcal{M} and over \mathcal{A} a differential calculus [5] $\{d, \Omega^*(\mathcal{A})\}$ which we shall choose so that it has as corresponding limit the ordinary de Rham

differential calculus. It is determined by the left and right module structure of the \mathcal{A} -module of 1-forms $\Omega^1(\mathcal{A})$. By definition a *metric* is a nondegenerate \mathcal{A} -bilinear map

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \mathcal{A}. \quad (1)$$

\mathcal{A} -bilinearity means

$$g(f\xi \otimes \eta h) = fg(\xi \otimes \eta)h, \quad (2)$$

for any $f, h \in \mathcal{A}$ and $\xi, \eta \in \Omega^1(\mathcal{A})$. It implies that g is completely determined by the \mathcal{A} -valued matrix elements

$$g^{ab} := g(\theta^a \otimes \theta^b), \quad (3)$$

for any choice of a basis of 1-forms $\{\theta^a\}$. In the commutative limit \mathcal{A} -bilinearity is equivalent to the very important requirement of locality of g in both arguments at each point $x \in \mathcal{M}$:

$$[g(f\xi \otimes \eta h)](x) = f(x) [g(\xi \otimes \eta)](x) h(x). \quad (4)$$

A *linear connection* is a map (see [14])

$$D : \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad (5)$$

together with a “generalized flip” σ , i.e. a \mathcal{A} -bilinear map

$$\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad (6)$$

going to the ordinary flip in the commutative limit and such that D satisfies the left and right Leibniz rules

$$D(f\xi) = df \otimes \xi + fD\xi \quad (7)$$

$$D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f. \quad (8)$$

Because of bilinearity, given a basis of 1-forms $\{\xi^i\}$ σ is completely determined once the \mathcal{A} -valued matrix elements S^{ij}_{hk} defined by

$$\sigma_0(\xi^i \otimes \xi^j) =: S^{ij}_{hk} \xi^h \otimes \xi^k \quad (9)$$

are assigned.

Let π be the projection

$$\pi : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A}). \quad (10)$$

The *torsion* is the map $\Theta = d - \pi \circ D$. In order that the torsion be bilinear we shall require

$$\pi \circ (\sigma + \mathbf{1}) = 0. \quad (11)$$

One can naturally extend D to higher tensor powers, e.g.

$$D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta), \quad (12)$$

where we have introduced the tensor notation $\sigma_{12} = \sigma \otimes \mathbf{1}$. The metric-compatibility condition for g, D reads $g_{23} \circ D_2 = d \circ g$.

The *curvature* $\text{Curv} : \Omega^1(\mathcal{A}) \rightarrow \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is defined by

$$\text{Curv} = \pi_{12} \circ D_2 \circ D. \quad (13)$$

It is always left \mathcal{A} -linear, and right \mathcal{A} -linear only in certain models; in general, right linearity is guaranteed only in the commutative limit. Therefore in this limit the curvature is local, an essential physical requirement for a reasonable definition of a curvature.

If $\mathcal{A}, \Omega^1(\mathcal{A})$ are $*$ -algebras and d is real, $(df)^* = df^*$, D is said to be real [13] if

$$D\xi^* = (D\xi)^* \quad (14)$$

where the involution on $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is defined by

$$(\xi \otimes \eta)^* = \sigma(\eta \otimes \xi^*), \quad (15)$$

with a σ such that the square of $*$ gives the identity. Note that this expression has the correct classical limit. So real structures on the tensor product are in one-to-one correspondence with right Leibniz rules. D_2 is real if D is and σ in addition fulfills the braid equation

$$\sigma_{12}\sigma_{23}\sigma_{12} = \sigma_{23}\sigma_{12}\sigma_{23}. \quad (16)$$

The curvature is real if D is real and (15), (16) are satisfied.

Now assume that there exists a *frame*, i.e. a special basis $\theta^a \in \Omega^1(\mathcal{A})$, $1 \leq a \leq n$, such that

$$[\theta^a, \mathcal{A}] = 0 \quad (17)$$

and any $\xi \in \Omega^1(\mathcal{A})$ can be uniquely written in the form $\xi_a \theta^a$, with $\xi_a \in \mathcal{A}$. This is possible only if the limit manifold \mathcal{M} is parallelizable. It has the advantage that for any $f \in \mathcal{A}$ the computation of commutator $[\xi, f]$ is

reduced to the computation of the commutators $[\xi_a, f]$ in \mathcal{A} . Assume also that there exist n inner derivations e_a ,

$$e_a f := [\lambda_a, f] \quad (18)$$

($\lambda_a \in \mathcal{A}$), dual to θ^a : $\theta^a(e_b) = \delta_b^a$. Then

$$\theta := -\lambda_a \theta^a \quad (19)$$

is the ‘Dirac operator’ [5] for d :

$$df = -[\theta, f]. \quad (20)$$

θ^a is a very convenient basis to work with. For instance, from \mathcal{A} -bilinearity it immediately follows that the corresponding elements (3) lie in the center $\mathcal{Z}(\mathcal{A})$ of \mathcal{A} , by the sequence of identities

$$fg^{ab} = g(f\theta^a \otimes \theta^b) = g(\theta^a \otimes \theta^b f) = g^{ab}f. \quad (21)$$

We shall be interested in the case that $\mathcal{Z}(\mathcal{A}) = \mathbf{C}$. In the commutative limit the condition $g^{ab} \in \mathbf{C}$ characterizes the vielbein or ‘moving frame’ of E. Cartan, which is determined up to a linear transformation; if this condition is fulfilled for any value of the deformation parameter the θ^a remain uniquely determined up to a linear transformation and are particularly convenient objects to be used to guess a physically sensible formulation of noncommutative-geometric notions.

From the above considerations we deduce that the metric is fixed by the form of the frame, and since the latter is fixed by the structure of the differential calculus up to a $GL(n)$ transformation, it will also be. Hence the differential calculus will yield a metric in the commutative limit as a *shadow of noncommutativity*. As a consequence, it seems that in this framework the metric cannot be considered as a dynamical variable, since it is completely determined by the differential calculus. This is completely different from what occurs in the commutative case, where the differential calculus and the metric on a smooth manifold are two independent structures. Something basically similar was proposed many years ago by Wheeler [16] when he suggested that the ‘graviton’ be considered as the ‘phonon’ of a fundamental space-time lattice. The fact that Wheeler was considering an ordinary lattice instead of a ‘quantum lattice’ is important of course but not essential.

One might be tempted to use this fact as an argument against noncommutative geometry. However one should note that in NCG the freedom lost in choosing the metric is recovered as a much larger freedom in choosing the differential calculus (without necessarily changing the commutative limit of the latter). In other words, the ‘degrees of freedom’ of the metric will be now encoded in the structure of the differential calculus.

2 Application of the formalism to the quantum Euclidean space

Take ‘the algebra of functions on the quantum Euclidean space \mathbf{R}_q^3 ’ [8] as \mathcal{A} and over it one of the two $SO_q(3)$ -covariant differential calculi [2]. The treatment of the other calculus can be done in a completely parallel way, see ref. [11]. Here we are interested in the case of a real positive q . We ask if they fit in the previous scheme. We shall denote by $\|\hat{R}_{hk}^{ij}\|$ the braid matrix of $SO_q(3)$, by $g_{ij} = g^{ij}$ the $SO_q(3)$ -covariant metric; here and below all indices will take the values $-, 0, +$. In the commutative limit $q \rightarrow 1$ $g_{ij} \rightarrow \delta^{i,-j}$. The projector decomposition of \hat{R} is

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{-2}\mathcal{P}_t; \quad (22)$$

$\mathcal{P}_s, \mathcal{P}_a, \mathcal{P}_t$ are $SO_q(3)$ -covariant q -deformations of respectively the symmetric trace-free, antisymmetric and trace projectors. The trace projector is 1-dimensional and is related to g_{ij} by

$$\mathcal{P}_{t\,kl}^{ij} \propto g^{ij} g_{kl}, \quad (23)$$

\mathcal{A} is generated by x^-, x^0, x^+ fulfilling $\mathcal{P}_a x x = 0$, or more explicitly

$$\begin{aligned} x^- x^0 &= q x^0 x^-, \\ x^+ x^0 &= q^{-1} x^0 x^+, \\ [x^+, x^-] &= h(x^0)^2. \end{aligned} \quad (24)$$

where we define $h = \sqrt{q} - 1/\sqrt{q}$. For real positive q the real structure on \mathcal{A} is defined by $(x^i)^* = x^j g_{ji}$, or more explicitly

$$(x^-)^* = \sqrt{q}x^+, \quad (x^0)^* = x^0, \quad (x^+)^* = 1/\sqrt{q}x^-. \quad (25)$$

$\mathcal{Z}(\mathcal{A})$ is generated by the $SO_q(3)$ -covariant real element

$$r^2 := g_{ij}x^i x^j = \sqrt{q}x^+ x^- + (x^0)^2 + 1/\sqrt{q}x^- x^+. \quad (26)$$

Let $\xi^i = dx^i$. One $SO_q(3)$ -covariant calculus, which we shall denote by $\{d, \Omega^*(\mathcal{A})\}$, is determined by the commutation relations

$$x^i \xi^j = q \hat{R}_{kl}^{ij} \xi^k x^l. \quad (27)$$

Unfortunately for real positive q neither calculus has a real exterior derivative, and up to now no way was known to make it closed under involution [18]; rather, each exterior algebra is mapped into the other under the natural involution. The ‘Dirac operator’ (20) corresponding to d is the $SO_q(3)$ -invariant element [21] $\theta := (q-1)^{-1} q^2 r^{-2} x^i \xi^j g_{ij}$; note that θ is singular in the commutative limit. As a contribution to the understanding of the structure of the quantum Euclidean spaces we have noticed [11] the following results

1. There exist two torsion-free, ‘almost’ metric-compatible linear connections, given by the formula

$$D_{(0)}\xi = -\theta \otimes \xi + \sigma_0(\xi \otimes \theta) \quad (28)$$

The two corresponding generalized flips σ_0 are the ones with matrix (9) given respectively by $S = q\hat{R}, (q\hat{R})^{-1}$. $D_{(0)}$ ‘almost’ metric-compatible means compatible up to a conformal factor with the metric which we shall give in the next item; a strict compatibility does not seem possible. Both σ_0 fulfill the braid equation (16) and both $D_{(0)}$ are $SO_q(3)$ -invariant.

2. If we extend \mathcal{A} by adding the ‘dilatation’ generator Λ

$$x^i \Lambda = q \Lambda x^i \quad (29)$$

together with its inverse Λ^{-1} (we shall normalize them so that $\Lambda^* = \Lambda^{-1}$) and set $d\Lambda = 0$, then up to normalization there exists a unique metric g_0 ,

$$g_0(\xi^i \otimes \xi^j) = g^{ij} r^2 \Lambda^2 \quad (30)$$

(g_{ij} is the $SO_q(3)$ -covariant metric matrix), which is compatible with the two $D_{(0)}$ up to the conformal factors q^2, q^{-2} ,

$$S^{ij}_{hk} g^{kl} S^{mn}_{jl} = q^{\pm 2} g^{im} \delta_h^n, \quad (31)$$

respectively in the cases $S = (q\hat{R})^{\pm 1}$. A strict compatibility would have required no $q^{\pm 2}$ at the rhs.

3. $\text{Curv}=0$ for both $D_{(0)}$.
4. If we further extend \mathcal{A} by adding also the generators r [the square root of (26)], its inverse r^{-1} and the inverse $(x^0)^{-1}$ of x^0 , then there exist a frame θ^a , $a = -, 0, +$, and a dual basis e_a of inner derivations given by

$$\theta^a := \Lambda^{-1} \theta_i^a \xi^i \quad (32)$$

with

$$\|\theta_i^a\| := \left\| \begin{array}{ccc} (x^0)^{-1} & & \\ \sqrt{q}(q+1)(rx^0)^{-1}x^+ & r^{-1} & \\ -\sqrt{q}q(q+1)(r^2x^0)^{-1}(x^+)^2 & -(q+1)r^{-2}x^+ & r^{-2}x^0 \end{array} \right\| \quad (33)$$

$$\begin{aligned} \lambda_- &= +h^{-1}q\Lambda(x^0)^{-1}x^+, \\ \lambda_0 &= -h^{-1}\sqrt{q}\Lambda(x^0)^{-1}r, \\ \lambda_+ &= -h^{-1}\Lambda(x^0)^{-1}x^-. \end{aligned} \quad (34)$$

$e_ax^i = q\Lambda e_a^i$, where $\|e_a^i\|$ is (left and right) inverse of the \mathcal{A} -valued matrix $\|\theta_i^a\|$. Its elements fulfill the ‘ RTT -relations’ [8]

$$\hat{R}_{kl}^{ij} e_a^k e_b^l = e_c^i e_d^j \hat{R}_{ab}^{cd} \quad (35)$$

as well as the ‘ gTT -relations’

$$g^{ab} e_a^i e_b^j = r^2 g^{ij} \quad g_{ij} e_a^i e_b^j = r^2 g_{ab}. \quad (36)$$

In a sense $r^{-1}e_a^i$ are a realization of the generators T_a^i of $SO_q(3)$. As a consequence we find

$$\mathcal{P}_{tcd}^{ab} \theta^c \theta^d = 0 \quad \mathcal{P}_{scd}^{ab} \theta^c \theta^d = 0, \quad (37)$$

the same commutation relations fulfilled by the ξ^i ’s. Similarly, the λ_i and the x^i satisfy the same commutation relations. Finally, up to a normalization $g_0(\theta^a \otimes \theta^b) = g^{ab}$.

5. $\Omega^*(\mathcal{A})$ is closed under the involution defined by

$$(x^i)^* = x^j g_{ji} \quad (\theta^a)^* = \theta^b g_{ba} \quad (38)$$

(the latter acts nonlinearly on the ξ^i ’s: $(\xi^i)^* = \Lambda^{-2} \xi^j c_{ji}$, with non-constant $c_{ji} \in \mathcal{A}$).

The reality structure of these differential calculi is an old, well-known problem (see [18]). The solution proposed in item 5 is not fully satisfactory, at least naively. For instance, it does not yield real d, D ; only the curvature is real, for the simple reason that it vanishes. The involution cannot be consistently extended to $\Omega^*(\mathcal{A}) \otimes \Omega^*(\mathcal{A})$ according to (15). Finally, apparently it has not the correct classical limit. Actually, the latter point can be solved by a more careful analysis [11] leading to the identification of x^i with some suitable general coordinates, as e.g. the ones reported at formulae (50) below.

A more careful analysis is needed at this point, but is out of the scope of the present report (for more details see Ref. [11]). It involves the investigation of the properties of the $*$ -representations of $\Omega^*(\mathcal{A})$ and seems to suggest a more sophisticated version of the proposal in item 5, in which the opposite properties of the two differential calculi cancel with each other. The problems mentioned above and the fact that the linear connections $D_{(0)}$ are metric-compatible up to conformal factors (or, in other words, are only conformally flat) may be related, in the sense that a satisfactory formulation of the reality properties could eventually yield also a new and satisfactory formulation of metric-compatibility which can be strictly fulfilled. A careful analysis of the commutative limit is also needed in order to propose a reasonable correspondence principle between the ‘new’ theory and classical differential geometry.

3 Representation theory and geometry

The set (x^i, e_a) generates the phase space algebra

\mathcal{D}_h of observables of a ‘point particle on \mathbf{R}_q^3 ’; we shall assume that x^i generate the subalgebra \mathcal{A}_p of position (i.e. configuration space) observables. In order to understand the geometrical structure of configuration space one should first consider the irreducible $*$ -representations of \mathcal{D}_h on Hilbert spaces H , and then attach to the configuration space observables x^i the appropriate physical meaning, with the help of the geometric tools (metric, curvature, etc) described in the previous sections. This will be done in detail elsewhere. Here we just give a flavour of how this may drastically change our naive expectations about the physical meaning of the generators x^i of \mathcal{A} , in the sense that they should be interpreted as a noncommutative generalization of generalized rather than cartesian coordinates on flat \mathbf{R}^3 . This will automatically cure some unpleasant features [12] which make the physical

interpretation of x^i as cartesian coordinates problematic, namely that within each irreducible $*$ representation the spectrum of x^0 has all eigenvalues of the same sign, that the value 0 is an accumulation point of the spectrum of $|x^0|$ on the left, whereas the difference between two neighbouring points of this spectrum diverges for $|x^0| \rightarrow \infty$.

A complete set of independent commuting observables (CSICO) must have three elements (three being the dimension of the classical underlying manifold). One cannot find three such elements within the subalgebra \mathcal{A}_p , since the latter is not abelian. As a CSICO we choose [12]

$$r, x^0, \mathbf{k}, \quad (39)$$

where \mathbf{k} can be identified with q^{L_0} and L_0 with an ordinary angular momentum component along the direction of an axis y^0 in ordinary Euclidean space (so the spectrum of L_0 is \mathbf{Z}). Assume that $q > 1$, for the sake of being specific. In Ref.'s [10, 12] it was shown that the irreducible $*$ -representations of \mathcal{D}_h for a zero spin point particle are essentially parametrized by a sign $\eta = \pm$ and a constant $c \in [1, q)$, and that for each (η, c) there exists a corresponding orthonormal basis $\{|n_0, n, m\rangle\}$ in which x^0, r are diagonal:

$$r|n_0, n, m\rangle = c q^n |n_0, n, m\rangle, \quad (40)$$

$$x^0|n_0, n, m\rangle = \eta c q^{(n-n_0-\frac{1}{2})} |n_0, n, m\rangle, \quad (41)$$

$$\Lambda|n_0, n, m\rangle = |n_0, n+1, m\rangle. \quad (42)$$

The integers are such that n_0 in $\mathbf{N} \cup \{0\}$ and m, n in \mathbf{Z} . m can be identified with the eigenvalue of the angular momentum component along a cartesian coordinate y^0 .

Here we do not write down the explicit action of the remaining generators of \mathcal{D}_h on this basis. We just note that by applying both sides of the identity $e_a f = [\lambda_a, f]$ to the generic vector $|\psi\rangle \in H$ one finds that it is consistent to set

$$e_a |\psi\rangle = \lambda_a |\psi\rangle, \quad (43)$$

where at the rhs the action of the element λ_a on H must be understood.

Assume for one moment that one could represent the exterior algebra $\Omega(\mathcal{A})$ on the same Hilbert space H (as we shall see, strictly speaking this is not possible). Since θ^a commutes with \mathcal{A} , from (43) we see that as an operator on H θ^a commutes with the whole \mathcal{D}_h ,

$$[\theta^a, \mathcal{D}_h] = 0. \quad (44)$$

Hence it is a *Casimir* of the representation:

$$\theta^a |\psi\rangle = t^a |\psi\rangle \quad (45)$$

for any $|\psi\rangle \in H$, where the objects t^a characterize H . Also wedge products of θ^a will commute with the whole \mathcal{D}_h , hence

$$\theta^a \theta^b |\psi\rangle = t^a t^b |\psi\rangle \quad (46)$$

$$\theta^a \theta^b \theta^c |\psi\rangle \equiv \epsilon^{abc} \theta^+ \theta^0 \theta^- = \epsilon^{abc} (t^+ t^0 t^-) |\psi\rangle \quad (47)$$

where ϵ^{abc} is the q -epsilon tensor [9]. Strictly speaking, one cannot take t^a in \mathbf{C} because then the t^a would commute with each other and therefore would not respect the commutation relations (37). One can however assume that at least $dv := t^+ t^0 t^-$ is a constant. Using the commutation relations and the values of ϵ^{abc} one can easily show that $dV := \theta^+ \theta^0 \theta^-$ is real w.r.t. the star structure \star ; so $dv \in \mathbf{R}$.

What is the physical interpretation of dv , the (unique) eigenvalue of the volume element observable dV ? It seems natural to consider it as the volume of the generic elementary cell in the configuration space

lattice. So the latter will be the same all over the configuration space. This is welcome because it means that the uncertainty in the localization of a point particle will be essentially the same all over the space.

One might argue that there can be no elementary cell in configuration space, since we cannot choose a CSICO consisting of three elements of \mathbf{R}_q^3 . With the choice (39) there are just elementary “annuli” V_{n,n_0} , defined by $cq^n \leq r < cq^{n+1}$, $q^{n_0} \leq \frac{|x^0|}{r} < q^{n_0+1}$. So in a sense we can just argue that the integral of dV on such annuli will give the volume of the latter. However this can be computed by a regularized trace on the eigenvectors of L_0 , by

$$V_{n,n_0} = C \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{m=-N}^N \langle n, n_0, m | dV | n, n_0, m \rangle = C \lim_{N \rightarrow \infty} dv = C dv \quad (48)$$

(C is an arbitrary normalization constant), which is also a constant. We shall now use this result to show that the generators x^i cannot go to cartesian coordinates on \mathbf{R}^3 in the commutative limit. In the commutative case the shell $R \leq r < R + \Delta R$, with r defined by $r = \sqrt{y \cdot y}$ and y^i cartesian coordinates, is a spherical shell in \mathbf{R}^3 , and therefore has a finite volume. On the contrary, in the present noncommutative space the shell $cq^n \leq r < q^{n+1}c$,

with r defined by $r = \sqrt{x \cdot x}$, will have an infinite volume, since it can be divided in an infinite number of annuli as above, with $n_0 \in \mathbf{N} \cup \{0\}$, each of which has the same volume. Obviously this result remains true when we take the commutative limit. Therefore in this limit the x^i might only go to some generalized, rather than cartesian, coordinates. Let

$$x^0 = f(\vec{y}) \quad r = g(\vec{y}) \quad (49)$$

be the transformation from some cartesian coordinates y^i on \mathbf{R}^3 to the commutative limit of the x^i . We can give a sense to this map also when $q \neq 1$, since we have assumed the commuting generators x^0, r to be position observables for any q . In Ref. [11] we have analyzed in some detail the constraints that a number of formal requirements puts on the commutative limit. One possible solution to these constraints gives for the functions f, g of (49)

$$\begin{aligned} x^0 &= e^{\alpha y^0 - \frac{\alpha^3}{2}} \\ r &= e^{-\alpha^3 + \frac{\alpha^2}{2} y^+ y^- + \alpha y^0} \end{aligned} \quad (50)$$

with $e^{\alpha^3} = q$. Thus, the surfaces $r = \text{const}$ are interpreted in physical space as paraboloids with axis y^0 rather than spheres with center in the origin, the surfaces $x^0 = \text{const}$ are interpreted in physical space as planes perpendicular to y^0 (exactly as before), the surfaces $x^0/r = \text{const}$ are interpreted in physical space as cylinders with axis y^0 rather than as cones centered at the origin, and the lines $x^0 = \text{const}, r = \text{const}$ are interpreted in physical space as circles perpendicular to and with center on the axis y^0 . The exponential relation between x^0 and y^0 is analogous to the one found [4] for a 1-dimensional q -deformed model. Due to the quantization of x^0, r , also y^0 and $y_\perp = \sqrt{y^+ y^-}$ are quantized. y^0 will take the values $y^0 = \alpha p$, whereas $y_\perp = \alpha \sqrt{n_0} + \frac{1}{2}$. So the eigenvalues of y^0 are equidistant and both positive and negative. Also, note that the step α goes to zero when $q \rightarrow 1$. Thus, the unpleasant features mentioned in the first paragraph of the present sections have been cured.

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